# Basic Parameters for Designing a "Macroscope"

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abstract
Several expressions for designing a "macroscope" are presented. Limitations on positions of mirrors are also discussed.

## 1. Introduction

In the preceding paper[1), the author has introduced a simple optical apparatus which enhances a 3-dimensional feeling for distant objects. Its structure is nothing different from an ordinary mirror-type stereoscope designed to view a large-sized pair of stereographs. What is new is its usage and scale. It is directed to a real object, a landscape for example, to stare it by naked eyes with effectively expanding their separation. I named this apparatus "macroscope".

In designing this apparatus it will be useful to provide some formulas to obtain the positions of mirrors and to optimize its main parameters, namely the angular field of view and the effective baseline length.

## 1. Definitions

Definitions of positions are illustrated in fig.1. Since the apparatus has a left-right symmetry, only the right half of its top view is shown. The center of the observer's eyes is taken to the origin of the Cartesian coordinate system. The position of the right eye is taken at ($a$,0). Objects are in the direction of the $y$-axis. The positions of the inner and outer edges of the "eye" mirror are denoted to P($x_1$, $y_1$) and Q($x_1'$, $y_1'$), respectively, and those of "objective" mirror are R($x_2$, $y_2$) and S($x_2'$, $y_2'$). For simplicity angle of these mirrors are fixed to 45 degrees to the $x$-axis. The position in the front-rear direction ($y$-axis) of the "eye" mirror is indicated by the intersection of the extension of the "eye" mirror plane with the $y$-axis, $y_0$.

Mathematics used is only an elementary one but the calculation is somewhat complicated and tedious.

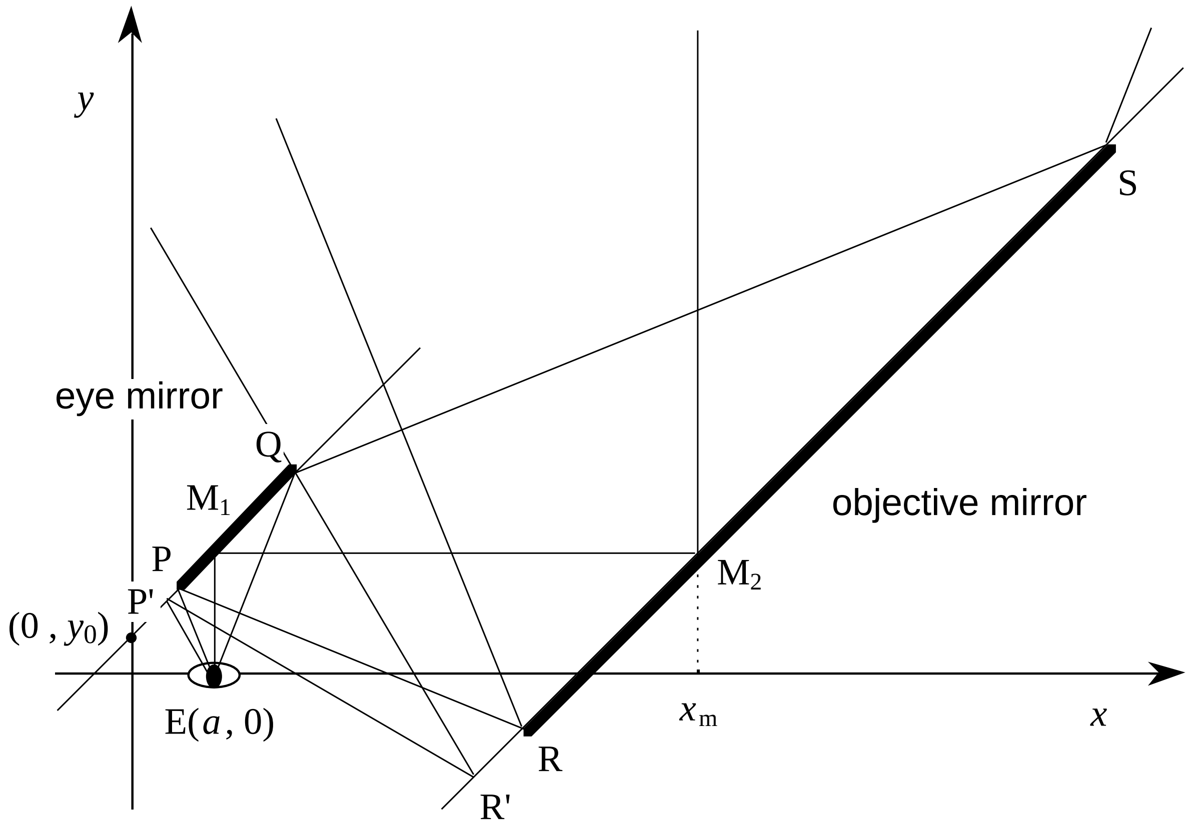



Fig.1. Right half of the apparatus is shown. E is the position of the right eye of the observer, $P(x_1,y_1)$ and $Q(x_1',y_1')$ are the edges of the "eye" mirror, $R(x_2,y_2)$ and $S(x_2',y_2')$ are those of "objective" mirrors. Angle of all mirrors are 45 degrees to the $x$-axis. $M_1$ and $M_2$ are positions where the central ray (parallel to the $y$-axis) is reflected.

## 2. Effective baseline length and horizontal angular field of view

Throughout this note we consider the horizontal size of the apparatus be a fixed parameter. The problem is to search an appropriate position (in the direction of $y$-axis) of two mirrors to give a wide angular field and largest possible baseline length. In other words, for a given $x_2'$, the change in the baseline length and the horizontal visual angle are examined when $y_0$ and $y_2'$ is varied. Once these parameters are determined, positions P, Q and R are calculated using the left-right symmetry of the visual angle.

Tangent of the right half of the visual angle α is

$$m = \frac{x_1' - a}{y_1'} \quad ....(1).$$

Half of the baseline length ($\overline{\mathrm{OX}}$ in fig.1) is proved to be $l = a + y_0 - y_2' + x_2'$ after a simple calculation. The dependence of $l$ on $y_0$ and $y_2'$ is obvious, but to know the dependence of $m$ on those parameters, we should calculate the position Q. This is readily known to be

$$\mathrm{Q}\left( -y_0 + \frac{(a+y_0)(x_2'+y_0)}{a+x_2'+2y_0-y_2'}, \frac{(a+y_0)(x_2'+y_0)}{a+x_2'+2y_0-y_2'} \right) \quad ....(2)$$

by using the fact that $\overline{\mathrm{EQ}}$ and $\overline{\mathrm{QS}}$ satisfies the reflecting condition with respect to the "eye" mirror and Q($x_1'$,$y_1'$) is on the line $y = x + y_0$. Inserting this into (1) we obtain

$$m = \frac{y_2' - y_0 - a}{x_2' + y_0} \quad ....(3).$$

The dependence of $l$ on $y_2'$ and $y_0$ is that the larger the $y_2'$, the smaller the baseline length, and contrary for $y_0$. The parameter $m$ is also nearly dependent on $y_0$-$y_2'$ because $x_2'$ is much grater than $|y_0|$, but the sign is reversed. Consequently these two parameters are in a competing relation; the larger the baseline length, the smaller the angular field. So one should find a compromised solution.

In choosing the parameter $y_0$, decreasing it reduces $l$ but increases $m$ . The latter effect is much more significant and one should choose the smallest value allowed by the geometrical interface to an observer's face.

Once a combination of $y_0$ and $y_2'$ is selected, the position P is determined by the condition that the left half of the visual angle is the same as the right one, owing to the left-right symmetry of the apparatus. Using (2), one finds

$$x_1 = \frac{ax_2' + 2ay_0 + y_0^2 - y_0 y_2'}{-a + x_2' + y_2'}, \quad y_1 = \frac{(a+y_0)(x_2'+y_0)}{-a + x_2' + y_2'}$$

and for R,

$$x_2 = \frac{ax_2' + x_2'^2 + 2ay_0 + 2x_2'y_0 + 2y_0^2 - x_2'y_2' - 2y_0y_2'}{-a + x_2' + y_2'} \ ,$$

$$y_2 = \frac{2ax_2' + 2ay_0 + 2x_2'y_0 + 2y_0^2 - ay_2' - x_2'y_2' - 2y_0y_2' + y_2'^2}{-a + x_2' + y_2'} \ .$$

## 3. Limiting conditions to the horizontal visual field

In the above calculation, the following limitations should be considered if large values of $y_2'$ or $y_0$ are met.

(1) Limitation of left(or inner) half of the visual angle due to the left edge of the right eye mirror which cannot cross over the $y$-axis.

(2) The innermost ray which is to be reflected at R and P must not be interrupted by the eye mirror.

The first condition means that the value $\frac{a}{y_0}$ should not be smaller than the right hand side of eq. (3). This immediately yields the following condition for $y_2'$ and $y_0$;

$$y_2' < \frac{ax_2' + 2ay_0 + y_0^2}{y_0} \qquad ....(4).$$

If a small value of $y_0$ is employed, this is automatically satisfied and this poses practically no limitation. As an example, for $a$ =3.2 cm and $x_2'$= 80.25cm, the inequality (4) means the lower region of the curve in the $y_0$-$y_2'$ plane, as shown in fig.2

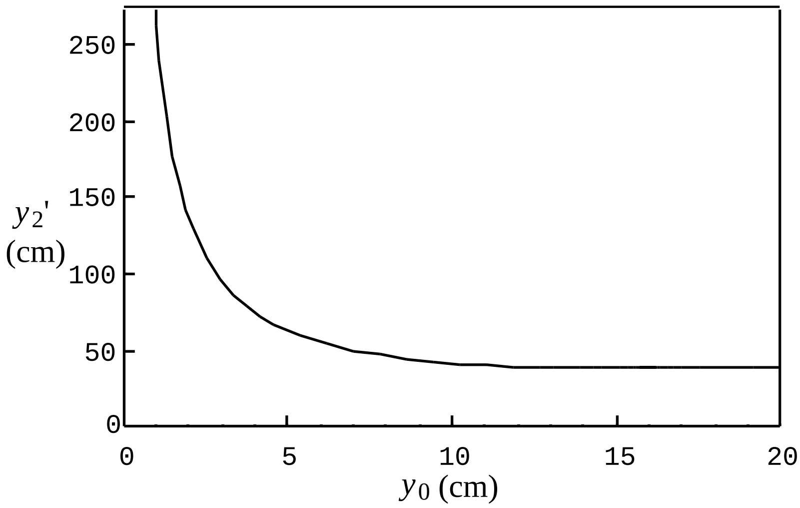
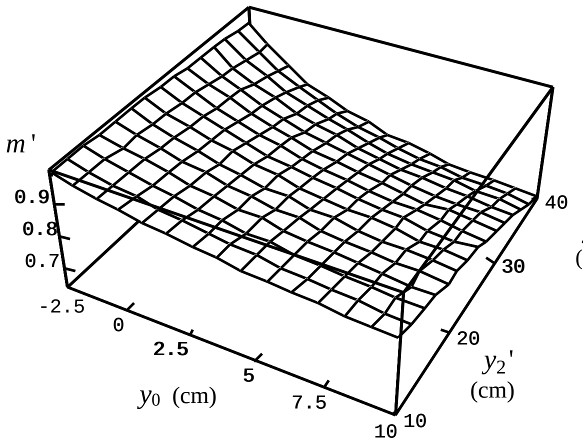



Fig.2. Limitation of inner half of the visual angle by the inner edge of the eye mirror for $a$ = 3.2cm and $x_2'$= 80.25cm. Below this curve is allowed.

To study the second condition, let us consider a ray which grazes the right edge(Q) of the eye mirror. This ray is reflected at R' and P', then arrive at the right eye. The condition is that the angle(denoted to α') of $\overline{\mathrm{EP'}}$ to the $y$-axis should be greater than or equal to that of $\overline{\mathrm{EQ}}$. Let $m'$ = tan α'. Using (2) and the fact that $\overline{\mathrm{EP'}}$ and $\overline{\mathrm{P'R'}}$ as well as $\overline{\mathrm{P'R'}}$ and $\overline{\mathrm{R'Q}}$ satisfies the reflection condition, $m'$ is known to be

$$m' = \frac{(a+x_2'+2y_0-y_2')^2-(a+y_0)(x_2'+y_0)}{(x_2'+y_0-y_2')(a+x_2'+2y_0-y_2')+(a+y_0)(x_2'+y_0)} \qquad ....(5).$$

For the above combination of values of $a$ and $x_2'$, the behavior of $m'$ is plotted in fig. 3. The condition $m$ =$m'$ is shown by a curve in fig. 4 and below this curve is the allowed area($m$ >$m'$). As seen from these figures, the second condition is satisfied unless very large $y_2'$ is chosen.



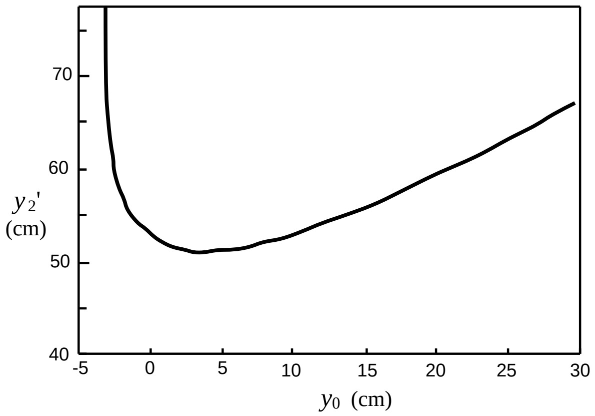




Fig.3(left). $m'$ is plotted against $y_0$ and $y_2'$ for $a$ = 3.2cm and $x_2'$= 80.25cm.
Fig.4(right). The region of $m>m'$ is below the curve.

## 4. Design for given widths of the apparatus and the objective mirrors

It would be useful to provide formulas for optimizing the mirror positions when the entire width of the apparatus and that of objective mirrors are given. In this case the task is to find out an optimum value of $y_2'$, and to calculate $x_1$, $y_1$, $x_1'$ and $y_1'$ for a given combination of $x_2'$ and $x_2$.

In section 1, positions of Q, P and R are already obtained as a functions of $a$, $y_0$ and the coordinates of S. Then $y_0$ is determined by use of the condition that $\overline{EP}$ and $\overline{EQ}$ makes the same angle with $\overline{EM_1}$, to be

$$y_0 = \frac{-a - x_2' + y_2' \pm \sqrt{-a + x_2' + y_2'}\sqrt{-a - \sqrt{2}W + x_2' + y_2'}}{2} \quad ....(6)$$

where $W$ is the horizontal width of the objective mirror. Negative sign before the square-root symbol should be discarded because it is an unphysical solution that the eye-mirror is put on the right to the eye, which means the staring line is directed to the $x$-axis. Inserting this expression into eq (2) we can get the expression for $m$;

$$m = \frac{-\left(a + \frac{W}{\sqrt{2}} - x_2'\right) t + (a - x_2') r}{\left(\frac{W}{\sqrt{2}} - y_2'\right) t - y_2' r} \quad ....(7).$$

The half of the baseline length $l$, on the other hand, is found to be

$$l = \frac{s + r}{2} \quad ....(8).$$

In these formulas the following replacements are employed to avoid lengthy expressions:

$$r = \sqrt{t(t-\sqrt{2}W)},\ \ s = a + x_2' - y_2',\ \ t = -a + x_2' + y_2'.$$

( Expression (6) is rewritten as $y_0 = \frac{-s+r}{2}$ with these replacements.)

An example of the behavior of $y_0$, $m$ and $l$ are illustrated in fig.5 for $a$ = 3.2cm, $x_2'$= 80.25cm and $W$ = 45cm. These graphs show that $y_2'$ is mostly determined to get a value of $y_0$ around zero.

## 5. Vertical sizes of mirrors

Visual field angle in the vertical direction is limited by the sizes of mirrors in that direction. Neccessary sizes of each edges, which give uniform vertical field, are obtained by proportionating a height at each edge to distances in $x$-coordinates from the image position ($-y_0$, $a+y_0$) of the right eye by the eye-mirror. If we denote a height of the edge S to $h_S$, heights at R, P and Q are as folows:

$$h_R = \frac{x_2 + y_0}{x_2' + y_0} h_S, \quad h_Q = \frac{x_1' + y_0}{x_2' + y_0} h_S, \quad h_P = \frac{x_1 + y_0}{x_2' + y_0} h_S .$$

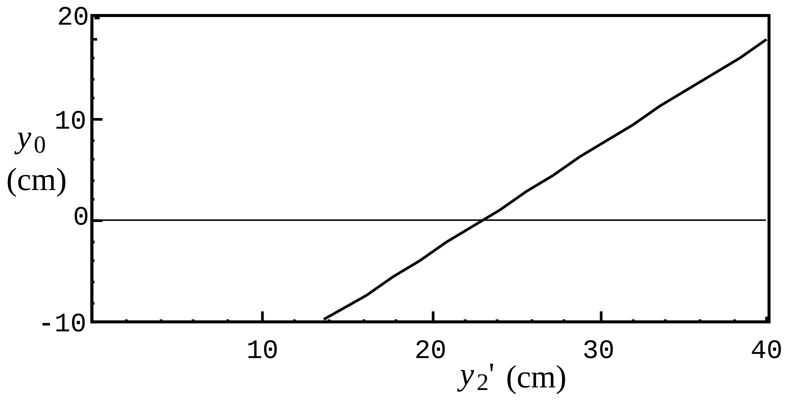



fig.5a

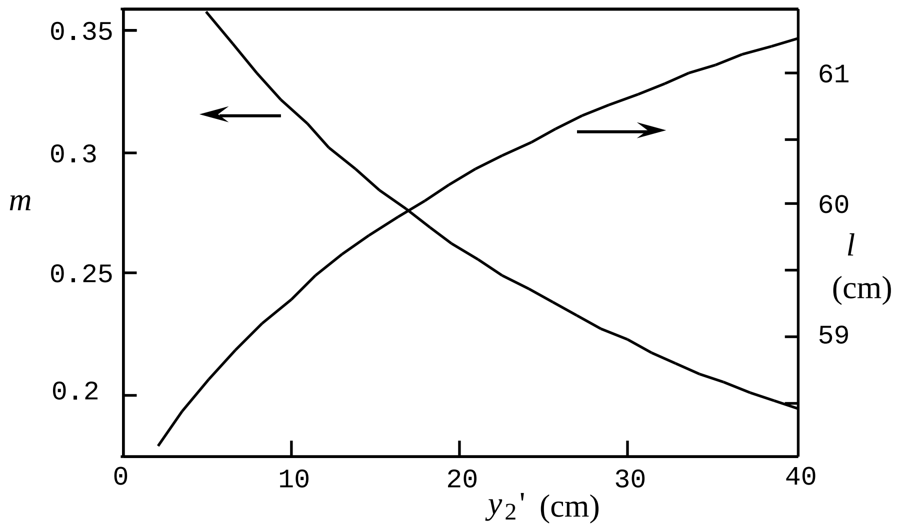



fig.5b

Fig.5. (a) Eye mirror position and (b) $m$, $l$ are plotted against $y_2'$ for $a$ = 3.2cm, $x_2'$= 80.25cm and $W$= 45cm.

I express my thanks to my sons Shinsaku and Kousaku who encouraged this work by hastening me to make up this apparatus.

Reference

[1)]K. Toyoshima, *Three-Dimensional Viewing of Distant Objects with "Naked" Eyes*.